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Some Sylow 2-Groups of Type $A \times B$, A Abelian

PAUL CHABOT

Department of Mathematics, California State University at Los Angeles, Los Angeles, California 90032

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1. INTRODUCTION

Let $P = A \times B$ be a Sylow 2-group of G, with A abelian and B nonabelian. In [3] we treated this situation with A elementary abelian and B dihedral, quasidihedral, or wreath product. The purpose of this paper is to formalize and generalize these fusion arguments.

In section 2 we establish some general lemmas which may also be of independent interest. In particular, Lemma 2.1 shows that P has a factorization in which both A and B are normal in $N(P)$, when B has no abelian direct factors. Assuming that P is in this form we define $i(P)$ to be the number of $N(P)$-conjugate classes of involutions in B. We treat the first few possibilities for $i(P)$.

All groups considered are finite. The notation is standard and may be found in [5].

We establish the following:

THEOREM 1.1. *Let $P = A \times B \in \mathrm{Syl}_2(G)$, where A is abelian and B has no abelian direct factors. Assume*

(i) $A \lhd N(P)$, $B \lhd N(P)$,

(ii) *If X is a $N(P)$-class in $\Omega_1(Z(P))$ and Y is a $N(P)$-class of involutions in B, then the product set XY is a $N(P)$-class,*

(iii) $\Omega_1(P') \leqslant Z(P)$,

(iv) $i(P) \leqslant 5$.

Then, $P = A_1 \times B$ with A_1 strongly closed in P with respect to G, except possibly when $Z(B)$ cyclic.

Note that $Z(B)$ cyclic implies that $\Omega_1(P')$ is cyclic. Thus, P' is either cyclic or generalized quaternion. In the case that P' is cyclic, [2] and [3] yield a result similar to that of Theorem 1.1 without assumptions (ii) and (iv). By considering the centralizers of elements of order 4 in P', it can be shown that the commutator group of a 2-group cannot be generalized quaternion.

Using the recent result of Goldschmidt [4], the conclusion of Theorem 1.1 may be strengthened when combined with the following:

PROPOSITION 1.2. *Let $P = A \times B \in \mathrm{Syl}_2(G)$, where A is abelian and strongly closed in P with respect to G. Assume $G = O^{2'}(G/O_{2'}(G))$. Then, $G = G_1 \times G_2$, where $G_1 = \langle A^G \rangle$ and $G_2 = \langle B_1{}^G \rangle$ for some B_1 with $P = A \times B_1$.*

2. LEMMAS

The following lemma is used to show that P has a direct decomposition $P = A \times B$, where both A and B are normal in $N(P)$. Since $N(P)$ is a split extension of P by a group T of odd order, this lemma concerns odd order action on a 2-group. Our proof uses only Maschke's theorem and thus also holds for p'-action on p-groups. We state the lemma in these terms.

LEMMA 2.1. *Let $P = A \times B$ be a p-group with A abelian and B nonabelian. Assume T is a p'-group acting on P. Then, $P = A_1 \times B_1$, where $A_1{}^T = A_1$, $B_1{}^T = B_1$, and B_1 has no abelian direct factors.*

Proof. Clearly we may assume $A \neq 1$ and B has no abelian direct factors. First we treat the case that $Z(P)$ contains a T-invariant subgroup $A_0 \neq 1$ with $A_0 \cap B = 1$. Let $\bar{P} = P/A_0$. By induction $\bar{P} = \bar{A}_1 \times \bar{B}_1$ where $\bar{A}_1{}^T = \bar{A}_1$ and $\bar{B}_1{}^T = \bar{B}_1$. Let A_1 be the full inverse image of $\bar{A}_1$. Let $P_1 = A_1\Omega_1(Z(P))$. By Maschke's theorem $\Omega_1(P_1) = \Omega_1(A_1) \times C_1$ with $C_1{}^T = C_1$. Moreover, $C_1 \lhd P$. We construct a sequence of C_i's. Let $P_{i+1} = A_1\Omega_1(Z(P \bmod C_i))$. Let $\tilde{P}_{i+1} = P_{i+1}/C_i$. By Maschke's theorem $\Omega_1(\tilde{P}_{i+1}) = \tilde{A}_1 \times \tilde{C}_{i+1}$ with $\tilde{C}_{i+1}^T = \tilde{C}_{i+1}$. Set C_{i+1} equal to the full inverse image of $\tilde{C}_{i+1}$. Finally, we set B_1 equal to the terminal member of the sequence of C_i's.

We complete the proof by demonstrating the existence of such a T-invariant subgroup A_0. Set $Z = \Omega_1(Z(P)) \cap \Phi(P)$, where $\Phi(P)$ is the Frattini subgroup of P. Note that $x \in \Omega_1(Z(P)) - Z$ implies that $\langle x \rangle$ is a direct factor of P. Let A_0 be a T-invariant complement to Z in $\Omega_1(Z(P))$. Since B contains no abelian direct factors of P, we see that $B \cap A_0 = 1$. Thus we are finished unless $A_0 = 1$. That is, $\Omega_1(A) \leqslant \mho^1(A)$. In this case, let C be a T-invariant complement to $P' \cap \Omega_1(Z(P))$ in $\Omega_1(Z(P))$. Set $\bar{P} = P/C$. By induction, $\bar{P} = \bar{B}_1 \times \bar{A}_1$, with $A_1{}^T = A_1$ and $B_1{}^T = B_1$. Let B_0 be the full inverse image of $\bar{B}_1$. We observe that $\Omega_1(Z(B)) \leqslant \Phi(B)$ and also that $\Omega_1(Z(B))$ is contained in any subgroup D of B such that $B = DZ(B)$. This allows us to see that the rank of $\Phi(B_0) \cap \Omega_1(Z(P))$ is equal to the rank of $\Omega_1(Z(B))$. Now let A_0 be a T-invariant complement to $\Phi(B_0) \cap \Omega_1(Z(P))$ in $\Omega_1(Z(P))$ and the proof is complete.

Most of the cases in Theorem 1.1 will be covered by the following lemma. One should note that assumption (ii)′ is a weakening of (ii).

LEMMA 2.2. *Let $P = A \times B \in \mathrm{Syl}_2(G)$, where $A \neq 1$ is abelian and B has no abelian direct factors. Assume*

(i) $A \lhd N(P)$, $B \lhd N(P)$,

(ii)′ *If X is a $N(P)$-class in $\Omega_1(A)$ and Y is a $N(P)$-class of involutions in B, then the product set XY is a $N(P)$-class,*

(iii) $\Omega_1(P') \leqslant Z(P)$.

Set n equal to the number of $N(P)$-classes of involutions in $A^{\#}$, m equal to the number of $N(P)$-classes of involutions in $B - Z(B)$, k equal to the number of $N(P)$-classes of involutions in $Z(B)^{\#}$. If $n + nk \geqslant nm + m$, then there exists a subgroup $A_0 \neq 1$ strongly closed in P with respect to G such that $A_0 \cap B = 1$.

Proof. $n + nk$ represents the number of mutually nonfused subsets of $\Omega_1(Z(P)) - \Omega_1(Z(B))$, while $nm + m$ is the number of $N(P)$-classes of involutions in $P - Z(P)$. If $\Omega_1(Z(B))$ is strongly closed in P with respect to G, then we argue that $\Omega_1(A)$ is also strongly closed in P with respect to G as follows. Let u be a 2-element in $N(P \cap Q)$ for some tame intersection $P \cap Q$ containing $Z(P)$. Let $a \in \Omega_1(A)$. Then, $[a, u] = 1$ or $[a, u]$ is an involutive commutator in the 2-group $(P \cap Q)\langle u \rangle$. Since $[a, u] \in P$ we see that $[a, u] \in \Omega_1(P') \leqslant Z(B)$. However, a is not fused to az for any $z \in Z(B)^{\#}$. Thus, u must centralize $\Omega_1(A)$. In view of Alperin's theorem [1] and the fact that $A \lhd N(P)$ we conclude that $\Omega_1(A)$ is strongly closed in P with respect to G.

If $\Omega_1(Z(B))$ is not strongly closed, then some element of $\Omega_1(Z(B))$ must be fused to one of the $nm + m$ $N(P)$-classes of involutions in $P - Z(P)$. Since $n + nk > nm + m - 1$, some $N(P)$-class X in $\Omega_1(Z(P)) - \Omega_1(Z(B))$ is not fused to anything in $P - Z(P)$. Let $W = \langle X \rangle$. Then, $W \leqslant \Omega_1(Z(P))$ is weakly closed in P. Thus, W is strongly closed since $W \leqslant Z(P)$. If $W \cap B = 1$, we are done. If $W \cap B = B_0 \neq 1$, then let A_0 be a $N(P)$-invariant complement to B_0 in W. Thus, A_0 is strongly closed in P with respect to G and $A_0 \neq 1$ since $W \nleqslant B$.

Those cases of Theorem 1.1 not covered by Lemma 2.2 will satisfy the assumptions of Lemma 2.3. That is, for the remaining cases $i(P) \leqslant 5$ will imply our assumption (iv)′.

LEMMA 2.3. *Let $P = A \times B \in \mathrm{Syl}_2(G)$, where A is abelian and B has no abelian direct factors. Assume*

(i) $A \lhd N(P)$, $B \lhd N(P)$,

(ii) *If X is a $N(P)$-class in $\Omega_1(Z(P))$ and Y is a $N(P)$-class of involutions in B, then the product set XY is a $N(P)$-class,*

(iv)′ *$N(P)$ transitive on $\Omega_1(P')^{\#}$.*

Then, either $\Omega_1(A)$ is strongly closed in P with respect to G or $|\Omega_1(P')| = 2$.

Proof. It should be noted that (iv)′ implies (iii) $\Omega_1(P') \leqslant Z(P)$. Let $Z = \Omega_1(P')^{\#}$. We first observe that (ii) and (iv)′ have the following consequence.

CLAIM 1. *$VZ = V$ for any $N(P)$-class of involutions V of $P - Z(P)$.*

By (ii) $V = XY$ where X is a $N(P)$-class in $\Omega_1(Z(P))$ and Y is a $N(P)$-class of involutions in B. Also, by (ii) $VZ = XYZ$ is a $N(P)$-class since YZ is a $N(P)$-class of involutions of B. Thus, it suffices to show that $VZ \cap V \neq \phi$.

Let $v \in V$ and take $w \in v^P - v$. If $\langle v, w\rangle$ is abelian, then for some $x \in P$ we have that $w = v[v, x] \in VZ$. But then $w \in VZ \cap V$ and the claim is established. If $\langle v, w\rangle$ is nonabelian, then let z be the involutive commutator of the dihedral group $\langle v, w\rangle$. We see that vz is a P-conjugate of v. Thus, $vz \in V \cap VZ$ and the claim is established.

Let u be a 2-element in $N(P \cap Q)$, where $Z(P) \leqslant P \cap Q$. Let $(P \cap Q)\langle u\rangle \leqslant R \in \mathrm{Syl}_2(G)$. Let $a \in \Omega_1(A)$ and assume that $|Z| > 1$.

CLAIM 2. *u centralizes a.*

Suppose $[a, u] \neq 1$. If $[a, u] \neq 1$. If $[a, u] \in Z(P)$, then $[a, u] \in Z$. This gives the contradiction that a is fused to az for some $z \in Z$. Thus, $[a, u] \in P - Z(P)$. Let $z \in Z$. If $[az, u] \neq 1$, then az is conjugate to $az[az, u]$. By Claim 1 applied to R, $az[az, u]$ is fused to $az[a, u]$. Now Claim 1 shows that $az[a, u]$ is fused to $a[a, u]$ which is clearly fused to a. Thus, since a is not fused to az, we must have that $[az, u] = 1$ for all $z \in Z$. Thus, u centralizes $z_1z_2 = (az_1)(az_2)$ for all $z_1, z_2 \in Z$. Since $|Z| > 1$ and $Z = \Omega_1(P')^{\#}$, u centralizes Z. Thus we may choose R so that $R \leqslant C(Z)$. But then we observe that $\Omega_1(R')^{\#} = Z = \Omega_1(P')^{\#}$. Thus, $[a, u] \in \Omega_1(R')^{\#} = Z \leqslant Z(P)$. This contradiction establishes Claim 2.

Since $A \lhd N(P)$, Alperin's theorem shows that $\Omega_1(A)$ is strongly closed in P with respect to G, as claimed.

3. Proofs

We shall carry out the main reduction of the theorem substituting the following weaker assumption for (ii).

(ii)′ *If X is a $N(P)$-class in $\Omega_1(A)$ and Y is a $N(P)$-class of involutions in B, then the product set XY is a $N(P)$-class.*

Assumption (ii)′ says that the $N(P)$ action on A is independent of the $N(P)$ action on B. For example, (ii)′ would be a consequence of:

(ii)″ *$C(B)$ controls fusion in A.*

Assumption (ii) contains also a restriction on the possible structure of B, especially when $i(P)$ is large.

Proof of 1.1. Clearly we may assume that $A \neq 1$. First, we treat the case that $Z(P)$ contains a subgroup $A_0 \neq 1$ which is strongly closed in P with respect to G and such that $A_0 \cap B = 1$. Let A_0 be a minimal such subgroup. Since $N(A_0)$ controls fusion in P, induction gives that $G = N(A_0)$. Examination of the proof of Lemma 2.1 shows that P has a direct factorization $A_1 \times B$, with $A_0 \leqslant A_1 \lhd N(P)$. We next argue that A may be replaced by A_1. This involves checking that (ii) hold for $A_1 \times B$. This is clear since (ii) is a statement about $Z(P)$ and B and does not depend on the complement to B. However, we shall show that if (ii)′ holds for $A \times B$, then (ii)′ holds for $A_1 \times B$.

Let $\Omega_1(A)^{\#} = X_1 \cup X_2 \cup \cdots \cup X_n$, where $X_i = x_i^{N(P)}$ and $X_i \cap X_j = \phi$ whenever $i \neq j$. Since $A_1 \lhd N(P)$, $\Omega_1(A_1)$ is a union of $N(P)$-classes. Let $Y = y^{N(P)}$ for some $1 \neq y \in \Omega_1(A_1)$. Assumption (ii)′ for $A \times B$ gives that $Y = X_i Z$, for some i and some $N(P)$-class Z of $\Omega_1(Z(B))$. If $z_1, z_2 \in Z$ then $x_i z_1, x_i z_2 \in X_i Z = Y \subseteq A_1$. Then, $z_1 z_2 \in A_1 \cap Z(B) = 1$ and $z_1 = z_2$. Thus $Y = X_i$ or $X_i z$ for some $1 \neq z \in Z(N(P))$. Now let W be any $N(P)$-class of involutions in B. Assumption (ii)′ for $A \times B$ yields that $X_i W$ is a $N(P)$-class. Thus, $X_i z W$ is also a $N(P)$-class. In either case we see that YW is a $N(P)$-class and that (ii)′ holds for $A_1 \times B$.

Now we may assume that $A_0 \leqslant A$. Let $\bar{G} = G/A_0$. Let C be a $N(P)$-invariant complement to A_0 in $\Omega_1(A)$ and let D be the full inverse image of $\Omega_1(\bar{A})$. Thus, $C \leqslant D$.

Case 1. $\bar{C} < \bar{D}$

Let $\bar{R}$ be a $N(P)$-invariant complement to $\bar{C}$ in $\bar{D}$ and set R equal to the full inverse image of $\bar{R}$. Thus, $D = R \times C$ and the minimality of A_0 gives that R is homocyclic with $\mho^1(R) = \Omega_1(R) = A_0$. Since squaring induces a natural $N(P)$-isomorphism from $\bar{R}$ to A_0, we see that the $N(P)$ action on

$\bar{R} \times \bar{C}$ is isomorphic to the action of $N(P)$ on $A_0 \times C$. Moreover, the action of $N(P)$ on $\bar{R} \times \bar{C} \times \bar{B}$ is isomorphic to the action of $N(P)$ on $A_0 \times C \times B$. Thus, $\bar{A} \times \bar{B}$ satisfies (ii)'. Induction on G/A_0 now completes the proof of the theorem in this case.

Case 2. $\bar{C} = \bar{D}$

This implies that $D = A_0 \times C$ and that A_0 is a direct factor of A. Since $\overline{N(P)} = N(\bar{P})$ and every $N(P)$-class of involutions of P is the image of a $N(P)$-class of involutions in $C \times B$, we can check that (ii)' holds for $\bar{A} \times \bar{B}$. Induction on G/A_0 now completes the proof in this case.

We note here that this reduction uses only (i) and (ii) and not (iii) nor (iv). The proof of the theorem will be completed by demonstrating the existence of such a subgroup A_0.

Since the number of $N(P)$-classes in $\Omega_1(Z(B))^{\#}$ must be odd, we see that Lemma 2.1 completes the proof except in the case that $N(P)$ is transitive on $\Omega_1(Z(B))^{\#}$ and $i(P) \geqslant 3$. But then $\Omega_1(Z(B))^{\#}$ must equal $\Omega_1(P')^{\#}$ and Lemma 2.2 can be applied to complete the proof.

Proof of 1.2. Goldschmidt [4] yields that $G_1 = \langle A^G \rangle = H_0 \times H_1 \times \cdots \times H_n$, where H_0 is a 2-group and H_i is a simple group with an abelian 2-Sylow, for $i \geqslant 1$. Moreover, $A \in \mathrm{Syl}_2(G_1)$.

If $H_0 \neq 1$, then $G = C(H_0)$ since $P \leqslant C(H_0)$ and $O^{2'}(G) = G$. Thus, $G = H_0 \times G_0$, for some subgroup G_0. If $H_0 = A$ we are done. If $H_0 < A$ then induction on G_0 will complete the proof.

Suppose $H_0 = 1$. Then, B normalizes each H_i since B centralizes $A \cap H_i \in \mathrm{Syl}_2(H_i)$. It is a well-known property of simple groups with abelian 2-Sylows that the only automorphism centralizing a 2-Sylow are inner. Thus, $B \leqslant G_1C(G_1)$. But $G_1C(G_1) = G_1 \times C(G_1)$. The proof is completed by taking $G_2 = C(G_1)$ and $B_1 = P \cap G_2$.

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